Abstract

In a previous article the transformation of space and time from an inertial system to another inertial system were obtained without using rigid measuring-rods and clocks as primitive entities. In this work the field equation is derived for the new transformation as well as the necessary mass for an object inside the black hole and in the neighborhood of the event horizon to reach the center (core) of a black hole. Using the new transformation the value of the vacuum energy density is found to be equal to $1.016124 \times 10^{113} \text{Jm}^{-3}$ and the contribution of the vacuum mass for the field equation is found to be $1.58 \times 10^{-26} \text{[kg m}^{-3}]$ for the Hubble parameter equal to $67.6 \text{[km s}^{-1} \text{Mpc}^{-1}]$.

1 Introduction

In Ferreira [1] transformation of space and time from an inertial system to another inertial system were obtained without using rigid measuring-rods and clocks as primitive entities. Section 2 shows the transformation obtained.

In sections 3 to 4 the energy-momentum tensor $T$ and the field equation are derived with the transformation obtained in [1]. The invariance of the speed of a particle in the transformation is used to obtain the energy-momentum tensor.

Sections 5 and 6 are on black holes. It is assumed that black holes have no singularities, instead they have cores. The transformation obtained in [1] is used in a black hole with zero angular moment to determine the time for a particle to go from the event horizon to the core of the black hole. It is assumed that a core has a radius equal to Planck's length in the center of the black hole. The necessary mass for an object inside the black hole and in the neighborhood of the event horizon to reach the core is given in section 6.

Weinberg [2] states that anything that contributes to the energy density of the vacuum acts just like a cosmological constant. In this work energy density of the vacuum and the cosmological constant are distinct things. In the section III of [2] we found:

$$|\lambda f| \leq H\theta^2$$

or

$$|\rho_\nu| \leq 10^{-26} \text{[kg m}^{-2}]$$

for a class of cosmological models described by Friedmann. Section 7 determines the value of the vacuum energy density, which is found to be equal to $1.016124 \times 10^{113} \text{Jm}^{-3}$. Sections 8 and 9 show that the value of the contribution of the vacuum mass density (denoted by $\Delta \rho_\nu$) for the field equation is not constant and calculates $\Delta \rho_\nu$ which is found to be $1.58 \times 10^{-26} \text{[kg m}^{-3}]$ for the Hubble parameter equal to $67.6 \text{[km s}^{-1} \text{Mpc}^{-1}]$. 

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2 The transformations

Let $S'$ be a reference system moving with respect to the reference system $S$ with constant speed $u$ in the direction $x^+$. The axis- $y'$ and the axis- $z'$ are parallel to axis- $y$ and axis- $z$, respectively. Let us suppose that the zero of $t'$ coincides with the zero of $t$ and the origin of $x', y', z'$ coincides with $x, y, z$ when $t = 0$. The Lorentz transformation is

$$ x' = \frac{x - ut}{(1 - \frac{u^2}{c^2})^{1/2}} $$

(3)

$$ y' = y $$

(4)

$$ z' = z $$

(5)

and

$$ t' = \frac{t - \frac{u^2}{c^2} x}{(1 - \frac{u^2}{c^2})^{1/2}} $$

(6)

The transformation found in Ferreira [1] is

$$ x' = (1 - \frac{u^2}{c^2})^{1/2} (x - ut) $$

(7)

$$ y' = (1 - \frac{u^2}{c^2})^{1/2} y $$

(8)

$$ z' = (1 - \frac{u^2}{c^2})^{1/2} z $$

(9)

and

$$ t' = (1 - \frac{u^2}{c^2})^{1/2} t $$

(10)

3 Energy-momentum tensor

In the special relativity the interval $s_{ij}$ between two events in system $S$ is defined by

$$ s_{ab}^2 = c^2 (t_b - t_a)^2 - (x_b - x_a)^2 - (y_b - y_a)^2 - (z_b - z_a)^2 $$

(11)

and $s_{ab}^2$ has the same value when calculated in the system $S'$. In the differential form (11) becomes

$$ ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 $$

(12)

where $x_0 = ct$, $x_1 = x$, $x_2 = y$ and $x_3 = z$

Let the relation between time $t$ in the system $S$ and time $t'$ in the system $S'$ be presented in [1] in the differential form

$$ dt' = \frac{dt}{\gamma} $$

(13)

the equation (13) can be rewritten as

$$ c^2 dt'^2 = c^2 dt^2 - (udt)^2 $$

(14)

for a particle traveling with constant velocity $u$ in $S$ (here we assume that $S'$ is moving in relation to $S$ with velocity $u$ and $u_1 = u_3$). Substituting $u$ by its components we have
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\[ c^2 \, dt'^2 = c^2 \, dt^2 - (u_i \, dt)^2 - (u_j \, dt)^2 - (u_k \, dt)^2 = c^2 \, dt^2 - (u_i \, dt)^2 \]  \hspace{1cm} (15) \]

Since

\[ u_i = \frac{dx_i}{dt} \quad \text{for} \quad i = 1, 2, 3 \]  \hspace{1cm} (16) \]

\[ c^2 \, dt'^2 = dx_0^2 \]  \hspace{1cm} (17) \]

and using (11) we have

\[ c^2 \left( \frac{dt}{\gamma} \right)^2 = (dx_0)^2 - (dx_i)^2 \]  \hspace{1cm} (18) \]

or

\[ ds^2 = \gamma (dx_0^2 - dx_i^2) \]  \hspace{1cm} (19) \]

Let be the cases in (12) where the differential interval components \( dx_2 \) and \( dx_3 \) are zero. Comparing (12) with (19) we see that with the transformation given in (13) other inertial observers will not find the same value for \( ds^2 \).

Let us consider the vector \( dx' = dx'_0, dx'_1, dx'_2, dx'_3 \), the time increment \( dt' \) and define the four-velocity vector

\[ U' = \frac{dx'_0}{dt'} \frac{dx'_1}{dt'} \frac{dx'_2}{dt'} \frac{dx'_3}{dt'} \]  \hspace{1cm} (20) \]

This vector is invariant in the proposed transformation in [1]. The vector \( U' \) can be rewritten as

\[ U' = (c, v) \]  \hspace{1cm} (21) \]

where

\[ v = \left( \frac{dx'_1}{dt'}, \frac{dx'_2}{dt'}, \frac{dx'_3}{dt'} \right) \]  \hspace{1cm} (22) \]

The mass \( m \) in the system \( S \) becomes \( \gamma m \) in the system \( S' \). Multiplying by \( \gamma mc \) both members of the module of (21) we obtain

\[ \gamma mcu' = (\gamma mc^2, \gamma mcv) \]  \hspace{1cm} (23) \]

or

\[ \gamma mcu' = (\gamma mc^2, p'c) \]  \hspace{1cm} (24) \]

Seeing that \( U' = U \) we have

\[ u'' = \eta_{\mu
u} u' \]  \hspace{1cm} (25) \]

\[ \eta_{\mu
u} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]  \hspace{1cm} (26) \]

Now consider the general case

\[ T_{\mu\nu} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix} \]  \hspace{1cm} (27) \]

In the system \( S' \) the volume contraction and the mass increase change the mass density
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\[ \rho = \gamma^4 \rho_0 \tag{28} \]

where \( \rho_0 \) is the mass density in the system \( S \). Since \( u^\nu - c \)

\[ T_{\mu\nu} = g \rho_0 c^2 = \frac{\rho}{\gamma} c^2 \tag{29} \]

and for \( T_{\mu\mu} \) with \( \mu \neq 0 \) we have

\[ T_{\mu\nu} = g \rho_0 (v^\nu)^2 = \frac{\rho}{\gamma^2} (v^\nu)^2 \tag{30} \]

where \( g \) will be given in section 4, for the time being \( g = 1 \). For a perfect fluid (a special case) there is no heat conduction (\( T_{0\nu} - T_{\mu0} = 0 \) for \( \mu \neq 0 \)) and the fluid has no viscosity (\( T_{\mu\nu} = 0 \) for \( \mu \neq \nu \)).

### 4 The field equation

The equation describing how matter generates gravity is modeled by

\[ G = kT \tag{31} \]

Provided that \( k \) is a scalar we have that \( G \) is a tensor of the same rank of \( T \) and symmetric. The tensor \( T \) has zero divergence to satisfy the law of conservation of momentum-energy

\[ \nabla \cdot T = 0 \tag{32} \]

Like \( T \) the tensor \( G \) must be a divergence-free tensor. When the space and time is flat the tensor \( G \) vanishes. Only the Riemann curvature tensor and metric construct \( G \) and the tensor \( G \) must be linear in the curvature. With these requirements \( G \) is unique, it is called the Einstein curvature tensor. The field equation then is [3].

\[ G = kT \tag{33} \]

We assume that \( k \) is a constant real number and that for \( v << c \) the equation (33) becomes the Newtonian gravity theory. Poisson’s equation for Newtonian gravity is

\[ \nabla^2 \Phi(x,t) = 4\pi G \rho(x,t) \tag{34} \]

where \( \Phi \) is the gravitational potential, \( G \) is the Newtonian gravitational constant and \( \rho \) is the mass density. The path of a free-falling particle for \( v << c \) satisfies the equation of motion

\[ \frac{d^2x(t)}{dt^2} = -\nabla \Phi(x,t) \tag{35} \]

Using (33), (34) and (35) we can determine the value of \( k \) and the eld equation becomes

\[ G = \frac{8\pi G'}{c^2} T \tag{36} \]

where

\[ T_{00} = \gamma^2 \rho_0 c^2 = \frac{\rho}{\gamma} c^2 \tag{37} \]

and for \( T_{\mu\nu} \) with \( \mu \neq 0 \) we have

\[ T_{\mu\nu} = \gamma^2 \rho_0 (v^\nu)^2 = \frac{\rho}{\gamma^2} (v^\nu)^2 \tag{38} \]
5 On black holes

Consider (i) a spheric object of mass \( M \) with its center in the origin of the system \( S \) and (ii) an observer stationary at a distance \( d \) from the object such that the gravitational attraction exercised by the object on the observer is negligible. The gravitational attractions on the object and the observer by other masses of the system \( S \) are negligible. There are no other masses for a distance from the mass \( M \) less than \( d \). In this case the metric is

\[
ds^2 = A(r,t)dt^2 - B(r,t)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]  

(39)

Let us consider radially moving particles

\[
d\theta = d\phi = 0
\]  

(40)

and we have

\[
ds^2 = A(r,t)dt^2 - B(r,t)dr^2
\]  

(41)

for a time-independent metrics we obtain

\[
ds^2 = A(r)dt^2 - B(r)dr^2
\]  

(42)

Around the object of mass \( M \) at a distance \( d \) the space is empty and \( T \) is zero. Therefore \( G \) is zero too. Since \( G \) is expressed in terms of the Ricci curvature tensor

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R
\]  

(43)

we have

\[
R_{\mu\nu} = 0
\]  

(44)

where

\[
R_{\mu\nu} = \Gamma^\rho_{\mu\rho\nu} - \Gamma^\rho_{\nu\rho\mu} + \Gamma^\rho_{\mu\nu} \Gamma^\rho_{\rho\mu} - \Gamma^\rho_{\nu\mu} \Gamma^\rho_{\rho\nu}
\]  

(45)

with

\[
\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^\rho_{\sigma\nu} (g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})
\]  

(46)

From (42) to (46) we can determine \( A(r) \) and \( B(r) \)

\[
A(r) = \mu (1 + \frac{k}{r})
\]  

(47)

and

\[
B(r) = (1 + \frac{k}{r})^{-1}
\]  

(48)

For a weak field

\[
A(r) = c^2 (1 + \frac{2\phi}{c^2})
\]  

(49)

Therefore

\[
A(r) = c^2 (1 + \frac{2GM}{c^2r})
\]  

(50)

\[
B(r) = (1 + \frac{2GM}{c^2r})^{-1}
\]  

(51)

and

\[
ds^2 = c^2 (1 + \frac{2GM}{c^2r})dt^2 - (1 + \frac{2GM}{c^2r})^{-1}dr^2
\]  

(52)

The (52) metric has a singularity at
The radius given in (53) is called Schwarzschild radius.

Now let us consider the object at the origin of the system $S$ at a time $t_0 - \Delta t$. Let us suppose that the mass of the object considered is $M - \Delta M$ at $t_0 - \Delta T$ and at the instant $t_0$ a mass $M$ has been integrated to the mass of the object. Let be $t_c >> t_o$. Let us suppose that the radius of the object of mass $M$ at $t_0$ is equal to the Schwarzschild radius, the mass density of the object is uniform and, immediately after $t_0$, when its volume is being collapsed, the mass density is uniform. The volume variation is given by

\[
dV_{sph}^\prime = \frac{dV_{sph}}{\gamma^3} \tag{54}
\]

The differential $dV_{sph}^\prime$ is in contact with the surface of the object. Since

\[
\frac{dV_{sph}^\prime}{dV_{sph}} = \frac{3/4}{3/4} = \frac{r^2}{r^2} \gamma R^2 \tag{55}
\]

we have

\[
\frac{1}{\gamma^3} = \frac{r^2}{R^2} \tag{56}
\]

where $R$ is the Schwarzschild radius of the object. From (56) and seeing that

\[
v = \frac{-dr}{dt} = \frac{-dr^\prime}{dt^\prime} = v^\prime \tag{57}
\]

we have

\[
v = c \sqrt{1 - \frac{r^2}{R^2}} \tag{58}
\]

and

\[
dt = -\frac{1}{c} \left(1 - \frac{r^2}{R^2}\right)^{1/2} dr \tag{59}
\]

The minus sign in (57) is because the differential volume moves inward

\[
\int_{s}^{t_{\text{Planck}}} -\frac{1}{c} \left(1 - \frac{r^2}{R^2}\right)^{1/2} dr = \frac{\pi}{2c} (R - t_{\text{Planck}}) \tag{60}
\]

The equation (60) gives the time for the contraction of the volume of mass $M$ for the observer in the system $S$. Also, this is the time spent for a particle to go from the event horizon to the distance to the center of the black hole equal to Planck’s length.

The time differential $dt^\prime$ in the volume $dV_{sph}^\prime$ is

\[
dt^\prime = -\frac{1}{c} \left(1 - \frac{r^2}{R^2}\right)^{1/2} \frac{dr}{\gamma} = -\frac{1}{c} \left(1 - \frac{r^2}{R^2}\right)^{1/2} \frac{R}{R} dr \tag{61}
\]

and the time in $S'$ for the contraction associated to the volume $dV_{sph}^\prime$ is

\[
\int_{s}^{t_{\text{Planck}}} -\frac{1}{c} \left(1 - \frac{r^2}{R^2}\right)^{1/2} \frac{R}{R} dr = \frac{1}{c} (R - t_{\text{Planck}}) \tag{62}
\]

The reason of the integration upper limit not to be zero is given in the next section.

6 The core of the black hole

Let $\gamma m_p$ be the mass of the particle, $\Delta r$ the distance of the particle to the center of
The black hole and

$$\Delta t = \frac{\Delta r}{c}$$  \hspace{1cm} (63)

Considering the Heisenberg principle

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$  \hspace{1cm} (64)

Let

$$(\gamma m_p c^2) \cdot (\Delta t) = \frac{\hbar}{2}$$  \hspace{1cm} (65)

Let us assume that the mass $M$ is composed of $n$ particles of mass $\gamma m_p$. In this case

$$Mc^2 = n\gamma m_p c^2$$  \hspace{1cm} (66)

and

$$\frac{Mc^2 \Delta t}{n} = \frac{\hbar}{2}$$  \hspace{1cm} (67)

$$\Delta r = \frac{n\hbar}{2Mc}$$  \hspace{1cm} (68)

Let us assume that (i) the minimum $\Delta r$ is equal to the Planck length $(1.61625281 \times 10^{-3}m)$, denoted by $l_{\text{planck}}$, and (ii) the core of the black hole is the volume with radius $l_{\text{planck}}$. Considering this, there is no singularity. The fall time from $R$ to $l_{\text{planck}}$ is

$$\frac{n\hbar}{2Mc}$$  \hspace{1cm} (69)

Comparing (60) with (69) we found

$$\pi(R - l_{\text{planck}}) = \frac{n\hbar}{2Mc}$$  \hspace{1cm} (70)

Let $M = 1.988 \times 10^{31} \text{kg}$ be the mass of a black hole (it is a mass ten times the mass of the Sun). In this case the value of $R$ is

$$R = 29525 \text{ m}$$  \hspace{1cm} (71)

Substituting in (68) $\Delta r$ by the Planck length

$$\frac{M}{n} = \frac{\hbar}{2cl_{\text{planck}}} = 1.088255 \times 10^{-4} \text{kg}$$  \hspace{1cm} (72)

Therefore the contraction from $R$ to $l_{\text{planck}}$ requires an object near to the surface of radius $R$ with mass equal to $1.088255 \times 10^{-4} \text{kg}$ (or $6.104659 \times 10^{18} \text{GeV}$) to reach the core. The ratio $M/n$ by the volume of the core is

$$\frac{3M}{8n\pi(l_{\text{planck}})^3} = 6.154000 \times 10^{10} \text{ Kg/m}^3$$  \hspace{1cm} (73)

and the energy density is

$$\frac{3Mc^2}{8n\pi(l_{\text{planck}})^3} = 5.530939 \times 10^{12} \text{ J/m}^3$$  \hspace{1cm} (74)

This suggests that the black hole masses are integer multiples of (73).

7 The vacuum energy

The classic Casimir result [4] for the force between perfectly conducting parallel plates is
where $\alpha$ is the distance between the plates. The difference of the energy density outside the plates and inside (between) the plates is

$$\Delta e = \left(\frac{\pi^2 \alpha c}{720 a^3}\right)$$

Let

$$a = \frac{l_{\text{Planck}}}{2}$$

be and consider areas in the plates, $\Delta A$, such that the energy contained in $\Delta A a$ is

$$\Delta E = \Delta e(\Delta A a) = \frac{\pi^2 \alpha c}{720 a^3} \Delta A$$

Let

$$\Delta A a = \frac{4}{3} \pi \left(\frac{a}{2}\right)^3$$

or

$$\Delta A = \frac{1}{6} \pi a^2$$

Thus

$$\Delta E = \frac{\pi^2 \alpha c}{720 a^3} \frac{1}{6} \pi a^2 = \frac{\pi^2 \alpha c}{4320 a}$$

and

$$\Delta e = \frac{\pi^2 \alpha c}{4320 a} \left(\frac{\pi a^3}{6}\right)^{-1} = \frac{\pi^2 \alpha c}{720 a^3}$$

or

$$\Delta e = \frac{\pi^2 \alpha c}{45 l_{\text{Planck}}^3}$$

or

$$\Delta e = 1.016124 \times 10^{13} \text{ J} \text{ m}^{-3}$$

The energy density of a black hole of mass $M$ minus the vacuum energy density is

$$n \times \frac{3 M c^2}{4 \pi n(l_{\text{Planck}})} = \frac{\pi^2 \alpha c}{45 l_{\text{Planck}}^3}$$

8 On the constancy of the vacuum energy

Let us consider the Friedman's equation

$$\frac{dR^2}{dt} = \frac{8 \pi G}{3} \rho R^3 - \frac{2E}{mx^2}$$

or

$$H^2 = \frac{1}{R^2} \frac{dR^2}{dt} = \frac{8 \pi G}{3} \rho - \frac{kc^2}{R^2}$$

where $\rho$ is the matter density inside a sphere of radius $r$ which is expanding about its center, $m$ is the mass of a particle on the surface of the sphere, $G$ is the Newtonian gravitational constant, $R$ is the scale factor, $H$ is the Hubble parameter, $k$ is a dimensionless constant related to the curvature of the universe, $c$ is the speed of light and $x$ is
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\[ x = \frac{r}{R} \]  

(88)

and the conservation equation is

\[ \frac{d\rho}{dt} - \frac{3}{R} \frac{dR}{dt} (\rho + \frac{p}{c^2}) \]  

(89)

the pressure \( p \) satisfies

\[ dE + pdV = 0 \]  

(90)

where \( V \) is the volume of the sphere. The equation (86) along with (88) governs the evolution of the scale factor of the universe.

Let us substitute \( \rho \) by \( \rho + \Delta\rho \) in (86)

\[ \frac{dR^2}{dt} = \frac{8\pi G}{3} (\rho + \Delta\rho) R^3 - \frac{2E}{mc^2} \]  

(91)

where \( \Delta\rho \) is the contribution of the energy density of vacuum, that is, \( \Delta\rho \) is part of \( \rho \).

The energy \( E \) is the total energy, including the contribution of vacuum density energy, and

\[ E = V (\rho + \Delta\rho) c^2 \]  

(92)

\[ V = \frac{4}{3} \pi R^3 \]  

(93)

Dividing (90) by \( c^2 dt \), we have

\[ \frac{1}{c^2} \frac{dE}{dt} + \frac{1}{c^2} \frac{dV}{dt} = 0 \]  

(94)

or

\[ \frac{dV}{dt} (\rho + \Delta\rho) + V \frac{d(\rho + \Delta\rho)}{dt} + \frac{1}{c^2} \frac{dV}{dt} = 0 \]  

(95)

Seeing that \( V \) is proportional to \( R^3 \)

\[ \frac{9}{16} \frac{dR}{dt} (\rho + \Delta\rho) + R \frac{d(\rho + \Delta\rho)}{dt} + \frac{9}{16} \frac{dR}{c^2 dt} = 0 \]  

(96)

or considering (87)

\[ \frac{1}{H} \frac{d(\rho + \Delta\rho)}{dt} = -\frac{9}{16} (\rho + \Delta\rho + \frac{P}{c^2}) \]  

(97)

but

\[ \frac{1}{H} \frac{d\rho}{dt} = \frac{1}{H} \frac{d}{dt} \left( \frac{r'}{1 + \frac{dr}{r}} \right) - 1 \rho \]  

(98)

\[ = \frac{1}{H} \frac{d}{dt} \left( \frac{1}{1 + \frac{dr}{r}} \right) - 1 \rho \]  

(99)

\[ = \frac{1}{H} \frac{d}{dt} \left( \frac{1}{1 - \frac{dr}{r}} \right) - 1 \rho \]  

(100)

\[ = \frac{1}{H} \frac{d}{dt} \left( \frac{1}{1 - 3\frac{dr}{r}} \right) - 1 \rho \]  

(101)
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\[ v = -3 \rho \frac{H}{r} = -3 \rho \quad (102) \]

therefore

\[ \frac{d \Delta \rho_v}{dt} = -9 \rho \frac{H + \Delta \rho_v + \frac{p}{c^2}}{16} + 3pH \quad (103) \]

or

\[ \frac{d \Delta \rho_v}{dt} = -9 \rho \frac{H + \Delta \rho_v + \frac{p}{c^2}}{16} + 3pH \quad (104) \]

From (105) we have that if \( \Delta \rho_v \) is constant with the time then

\[ p = c^2 (-\Delta \rho_v + \frac{39}{9} \rho) \quad (105) \]

and the pressure \( p \) becomes negative when

\[ \rho < \frac{9}{39} \rho \Delta \rho_v \quad (106) \]

### 9 The value of \( \Delta \rho_v \)

Let us consider a control volume, denoted by \( cv_1 \), with volume \( V \) which has energy density equal to \( \rho_v \) and is moving with velocity \( v \) in system \( S \). This control volume is stationary in the system \( S' \). In \( S' \) the volume is contracted to

\[ v' \frac{V'}{V} = (1 - \frac{v'}{c})v'^2V \quad (107) \]

The mass of the control volume in \( S \) is

\[ m = \rho_v V \quad (108) \]

and assuming that in \( S' \) the mass is

\[ m' = \gamma \rho_v V \quad (109) \]

we obtain

\[ \rho_v' = \frac{\rho_v}{\gamma^2} = (1 - \frac{v'}{c})\rho_v \quad (110) \]

Now let us assume the control volume, denoted by \( cv_2 \), is at the distance \( r \) of the origin of \( S \) and moving away at the velocity given by

\[ v = Hr \quad (111) \]

where \( H \) is the Hubble parameter. Let \( v \) found in (107) be equal to \( v \) given by (111) by hypothesis. So the mass density of the volume control \( cv_2 \) is equal to

\[ \rho_v' = (1 - \frac{HR}{c^2}) \rho_v \quad (112) \]

Let us complete this set of hypotheses assuming

\[ \Delta \rho_v = \rho_v - \rho_v' = \frac{HR}{c^2} \rho_v \quad (113) \]

and
\[ r = l_{\text{Planck}} \] (114)

In the literature we can find \( H_0 \) varying between \((63.7 \pm 2.3) \text{ km s}^{-1} \text{Mpc}^{-1}\) and \((74.7 \pm 2.6) \text{ km s}^{-1} \text{Mpc}^{-1}\) \([5]\). Using \( \Delta e = 1.016124 \times 10^{13} \text{ [J m}^{-3}\text{]}\) and \( H = 67.6 \text{ [km s}^{-1} \text{Mpc}^{-1}]\) we obtain

\[
\Delta \rho_v = \left( \frac{H_{\text{Planck}}}{c} \right)^2 \frac{\Delta e}{c^2} = 1.58 \times 10^{-28} [\text{kg m}^{-3}]
\] (115)

References